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On domain shapes and processes in supersymmetric theories

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Abstract

A supersymmetric theory with several scalar superfields generically has several domain wall type classical configurations which interpolate between various supersymmetric vacua of the scalar fields. Depending on the couplings, some of these configurations develop instability and decay into multiple domain walls, others can form intersections in space. These phenomena are considered here in a simplest, yet non-trivial, model with two scalar superfields.

1 Introduction

Domain walls in supersymmetric theories have attracted some attention in recent literature both in a general theoretical aspect[1] and in relation to a possible phenomenology of the early Universe[2]. Supersymmetric theories may posses a rich non-trivial structure of domain walls. Indeed, in a renormalizable 4-dimensional theory with N chiral superfields Φ_i ($i = 1, \ldots, N$) a generic superpotential $W(\Phi_i)$ is a cubic polynomial of the fields. The supersymmetric vacua of the scalar fields ϕ_i are determined by the equations $\partial W(\phi_i)/\partial\phi_i = 0$. These equations in general have 2^N solutions (with generally complex fields ϕ_i), each corresponding to a vacuum state v_a ($a = 1, \ldots, 2^N$) with zero energy. Accordingly, there are $2^{N-1}(2^N - 1)$ domain wall configurations, each described by a solution to the classical field equations that depends on only one coordinate (z) and interpolates between different vacua v_a and v_b at two different infinities in z, a configuration which here will be referred to as an "ab wall" or w_{ab}[1] Depending on the couplings between the fields there can be various relations between the energies of these configurations. In particular, if there is a vacuum state v_c such that the energy of the ab wall, ε_{ab}, is larger than the sum of the energies of the walls ac and cb: $\varepsilon_{ac} + \varepsilon_{cb}$, then the ab wall would decay into the pair $w_{ac} + w_{cb}$. In other words instead of a well localized in z transition between the domains with the vacua v_a and v_b there arises an arbitrarily large domain of the vacuum v_c, so that the localized transitions are $v_a \to v_c$ and $v_c \to v_b$. The latter picture can also be characterized as a stratification of a two-phase configuration (v_a, v_b) into a three-phase one (v_a, , v_c, v_b) This process immediately invites the problem of whether the wall w_{ab} is metastable or absolutely unstable, i.e. whether there is or there is not an energy barrier separating the configuration w_{ab} from the $w_{ac} + w_{cb}$.

Another problem, related to the multitude of vacua in supersymmetric theories, arises if one considers boundary conditions in the 3 dimensional space, involving more than one coordinate, namely with different domains at different directions in space. Then the boundaries between the domains have to intersect in space, thus suggesting the problem of the stability of the intersections, and of the shapes formed by the domain walls at the intersection. (It is known since long ago that an intersection of domain walls in a theory of one scalar field is not stable.)

A complete consideration of these problems looks rather complicated even in a theory

[1]It should of course be understood that additional constraints, including the gauge symmetry constraints, do restrict the form of the superpotential and can reduce the number of possible vacua.

with two scalar fields (four vacua) if the superpotential is assumed to be of a generic form of a cubic polynomial. In this paper these problems are addressed in a recently considered[2] simplified case of a particular superpotential and partial results are obtained illustrating the dependence of the stability of certain wall configurations on the couplings in the model. The superpotential chosen here for consideration is

$$W(\Phi,\, X) = \lambda\, X\, (\Phi^2 - a^2) + \frac{1}{3}\, \mu X^3 \tag{1}$$

with Φ, X being the superfields, λ and μ being dimensionless couplings, and, finally, a being a dimensionful parameter. The potential V for the scalar components ϕ and χ of the superfields

$$V(\phi,\, \chi) = \left|\lambda\, (\phi^2 - a^2) + \mu\, \chi^2\right|^2 + 4\, |\lambda\, \phi\, \chi|^2 \tag{2}$$

has four supersymmetric minima[2] : $\phi = \pm a,\ \chi = 0$ and $\phi = 0,\ \chi = \pm\sqrt{\lambda/\mu}\, a$. These vacuum states are ascribed here numbers 1 through 4 as shown in Figure 1. Correspondingly there are 6 domain wall configurations w_{ab} connecting pairs of the vacua.

Due to the $Z_2 \times Z_2$ symmetry of the potential under reversing the sign of either ϕ or χ the energies of the four domain walls w_{12}, w_{23}, w_{34} and w_{41} are degenerate, while the energies of the rest two wall configurations,the "diagonals" (in Fig.1) w_{13} and w_{24} are generally different. It is found here that depending on the ratio of the coupling constants $\xi^2 = \mu/\lambda^{\,2}$ at least one of the latter configurations can be unstable. Namely, for $\xi > 1$ the w_{13} decays into $w_{12} + w_{23}$ or $w_{14} + w_{43}$, while the "diagonal" w_{24} is stable at least locally, i.e. with respect to small perturbations at all values of ξ. The "diagonal" w_{13} is stable locally for all $\xi < 1$ and is also stable globally at small $1 - \xi$. For $\xi = 1$ the energy of each "diagonal" wall is exactly equal to the sum of the energies of two "sides", so that finer effects, than considered here, determine the (in)stability of the "diagonals". It is also found here that the instability of the w_{13} at $\xi > 1$ is absolute in the sense that the spectrum of small excitations around the wall configuration develops a runaway mode. Thus the unstable wall is not a metastable local minimum of the energy in space of field configurations interpolating between two corresponding vacua[3] . Needless to mention that the particular superpotential in eq.(1) represents only a very limited class of possible models, thus one can only hypothesize on the generality of this behavior.

[2]It is sufficient to consider only the case of positive μ/λ, since the relative sign of the couplings can be reversed by relabeling χ as $i\,\chi$.

[3]The results in this paper also illustrate that the local instability of the "diagonal" wall w_{13} at $\xi > 1$ does not lead to the so-called ribbon configurations[2] inside the wall, but rather results in a complete decay of the wall into two stable walls.

The relations between the surface energies of the domain walls also determine the possible shapes of intersections of the domain boundaries in the cases where the boundary conditions at the space infinity require presence of more than just two vacuum phases. We will show that the equilibrium shapes of the intersections of the walls are determined by the stratification considerations and by simple equations for the intersection angles, analogous to those in the capillarity theory.

2 Global stability and instability of "diagonal" walls

Throughout this paper the parameters of the superpotential λ, μ and a are assumed to be positive real. Then the vacuum states, the field profiles, and also possible runaway modes are all described by real fields ϕ and χ.

The profile of the "diagonal" walls (positioned across the z axis and centered at $z = 0$) is given by the familiar one-field formulas:

$$w_{13}\ : \qquad \phi(z) = a\ \tanh(\lambda\, a\, z)\ , \quad \chi(z) = 0\ ; \tag{3}$$

$$w_{24}\ : \qquad \chi(z) = \frac{1}{\xi}\, a\ \tanh(\xi\, \lambda\, a\, z)\ , \quad \phi(z) = 0\ . \tag{4}$$

The surface energy densities of these walls are respectively $\varepsilon_{13} = \frac{8}{3}\,\lambda\, a^3$ and $\varepsilon_{24} = \frac{8}{3}\,\lambda\, a^3/\xi$.

The profiles of all four "side" walls are related by the $Z_2 \times Z_2$ symmetry of the model considered here and they have same energy. Thus it would be sufficient to find the profile and the energy density of one of these walls, e.g. w_{12}. However, an analytical solution is not readily available, except for the case of $\xi = 1$, where an additional symmetry arises with respect to the permutation $\phi \leftrightarrow \chi$. In this case the problem is reduced to the one-field domain wall, by a $\pi/4$ rotation in the field space, and the solution is

$$w_{12}\,(\xi = 1)\ : \qquad \phi = \frac{a}{2}\,[1 - \tanh(\lambda\, a\, z)]\ , \quad \chi = \frac{a}{2}\,[1 + \tanh(\lambda\, a\, z)]\ . \tag{5}$$

The energy density of this configuration is $\varepsilon_{12}(\xi = 1) = \frac{4}{3}\,\lambda\, a^3$, thus a two "side" wall configuration (e.g. $w_{12} + w_{23}$) has the same energy as a "diagonal" wall (w_{13}) in the limit where the two side walls are separated by large (formally infinite) distance, so that their interaction is neglected.

In order to assess the relation between the sum of the energies of the "side" walls and of the "diagonal" one for $\xi \neq 1$ two approaches are used here: perturbation over

the configuration in eq.(5) for small $|\xi - 1|$ and a variational bound for the energy of the "side" walls at arbitrary ξ.

In order to apply the perturbation theory we write the potential in the sector of real fields as a function of the fields ϕ and χ and of the parameters λ and ξ:

$$V(\phi,\, \chi,\, \lambda,\, \xi) = \lambda^2 \left[(\phi^2 + \xi^2\, \chi^2 - a^2)^2 + 4\, \phi^2\, \chi^2 \right] \tag{6}$$

and notice that from dimensional considerations the energy density of a wall depends on the parameters as $\varepsilon = \lambda\, a^3\, f(\xi)$ with the non-trivial information contained in the dimensionless function $f(\xi)$. Thus one can apply the standard perturbation theory formula for the dependence of the energy density ε_{12} on ξ at $\xi = 1$ and find:

$$\frac{d\varepsilon_{12}}{d\xi} = \int \left. \frac{\partial V(\phi,\, \chi;\, \lambda,\, \xi)}{\partial \xi} \right|_{\phi=\phi_0,\, \chi=\chi_0,\, \xi=1} dz = -\frac{4}{3}\, \lambda\, a^3 \;, \tag{7}$$

with ϕ_0 and χ_0 given by the exact solution (5) at $\xi = 1$. Comparing the energy of the "side" walls in this order in $|\xi - 1|$ with the energy of the "diagonal" walls described by the equations (3) and (4), one finds that the wall w_{13} is stable with respect to decay into two "side" walls at $\xi < 1$ and is unstable at $\xi > 1$. Naturally, this conclusion is valid only in a finite region of ξ near $\xi = 1$. It will be shown by a variational bound that the instability of the w_{13} holds for all ξ greater than one. In the whole domain $\xi < 1$, strictly speaking, only the local stability of the wall w_{13} will be established here. However it would be quite surprising if there is a sub-domain of ξ at $\xi < 1$ where the global stability of the wall w_{13} is broken. As to the "diagonal" wall w_{24}, its energy coincides with the sum of the energies of two "side" walls in this order in $|\xi - 1|$ and no conclusion about its global stability can be made in this order. We shall see however that the wall w_{24} is locally stable for all values of ξ at least up to loop effects.

In order to obtain a variational bound for the energy density of the "side" wall w_{12} a 'good' trial configuration should be constructed interpolating between the vacua v_1 and v_2. We use here the obvious property of the potential V that it retains its quartic form under a linear transformation of the fields. Thus the potential has a quartic dependence along the straight line in the field space connecting the vacua v_1 and v_2 in which vacua the potential along the line has degenerate minima. Therefore if the trajectory of the trial configuration in the field space is restricted to this line, the minimal energy is given by the standard solution for the one-field wall. This solution for the trial configuration is readily found as

$$\phi_t(z) = \frac{a}{2}\, [1 - \tanh(\lambda\, a\, z)] \;, \qquad \chi_t(z) = \frac{a}{2\, \xi}\, [1 + \tanh(\lambda\, a\, z)] \;. \tag{8}$$

The energy of this configuration is given by $\varepsilon_t = \frac{2}{3}\,(1+\xi^{-2})\,\lambda\,a^3$, which is our upper bound for the actual value of the energy of any of the "side" walls. One can notice that this upper bound is quite 'good' in the sense that not only it reproduces the actual energy at $\xi = 1$ (as it should by construction) but also reproduces the slope of the energy at $\xi = 1$.

One can now see that at $\xi > 1$ the energy of the "diagonal" wall w_{13} always exceeds $2\,\varepsilon_t$. Thus the wall w_{13} is unstable with respect to decay into a pair of "side" walls. As to the energy of the other "diagonal", w_{24}, it is not greater than $2\,\varepsilon_t$ at any ξ, and this trial configuration is of little use for determining the stability of the w_{24}[4].

3 Local stability consideration

The stability of the walls with respect to small perturbations is analyzed by the standard method of linearizing the field equations near the classical solution and finding the spectrum of the eigenvalues for ω^2 with ω being the frequency of the mode: $\psi(t,x,y,z) = e^{-i\omega t}\,u(x,y,z)$. Negative eigenvalues for ω^2 correspond to runaway modes and imply the instability of the configuration around which the mode expansion is performed. Conversely, absence of negative modes means that the classical background is stable at least locally, i.e. with respect to small perturbations.

The linearized equations have the form of the Schrödinger equation in which the classical background provides the potential that depends on z under our convention about the placement of the walls. Negative modes arise when the potential is sufficiently negative over a sufficiently wide region of z. With real coupling constants and in a real background, the energy of the modes in the imaginary direction of the fields is always positive definite with the potential given by eq.(2). Thus in a search for negative modes one can disregard the perturbation of the fields in the imaginary direction and consider only real fields.

The complete field equations in the model considered have the form

$$\partial^2\,\phi + 2\,\lambda\,\phi\left[\lambda\,(\phi^2 - a^2) + \mu\,\chi^2\right] + 4\,\lambda^2\,\phi\,\chi^2 = 0\;, \tag{9}$$

$$\partial^2\,\chi + 2\,\mu\,\chi\left[\lambda\,(\phi^2 - a^2) + \mu\,\chi^2\right] + 4\,\lambda^2\,\chi\,\phi^2 = 0\;, \tag{10}$$

where $\partial^2 = \partial_t^2 - \nabla^2$. Let us first consider the perturbations over the wall w_{13} for which the classical solution is given by eq.(3). Since in this background $\chi = 0$, the linearized

[4] As well as the whole variational approach is not very helpful here, since the wall w_{24} is most likely globally stable at all ξ.

equations for the ϕ-modes and the χ-modes decouple: the former modes are described by linearization of the equation (9) in perturbation of ϕ after χ is set identically equal to zero, while the latter modes are described by the linear in χ part of the equation (10) with the ϕ replaced by the classical background. Thus the problem for the ϕ-modes reduces to the standard one-field situation in which it is known since long ago[3, 4] that the lowest mode is the zero one and there are no negative modes. Therefore a negative mode can arise only from the linearized equation (10) for the χ-modes:

$$\partial^2 \chi + \lambda^2 a^2 \left(4 - \frac{4 + 2\,\xi^2}{\cosh^2 \, \lambda \, a \, z} \right) \chi = 0 \; . \tag{11}$$

The lowest eigenvalue in this well known Quantum Mechanical problem is given by

$$\omega^2_{min} = \lambda^2 \, a^2 \, \left[4 - \frac{1}{4} \left(\sqrt{17 + 8\,\xi^2} - 1 \right)^2 \right] \; . \tag{12}$$

One can readily see from this expression that ω^2_{min} becomes negative for $\xi > 1$. Thus the wall w_{13} is unstable locally at $\xi > 1$ and is stable under small perturbations at $\xi < 1$.

For the other "diagonal" wall, w_{24}, described by eq.(4) the meaning of the ϕ and χ modes is reversed: the linearized equation (10) describes the χ modes in a situation equivalent to the one-field problem, while the linearized equation (9), describing the ϕ modes should be inspected for a possible presence of negative eigenvalues. The linearized equation for the ϕ modes has the form

$$\partial^2 \phi + \xi^2 \, \lambda^2 \, a^2 \, \left(\frac{4}{\xi^4} - \frac{4 + 2\,\xi^2}{\xi^4 \, \cosh^2 \, \xi \, \lambda \, a \, z} \right) \phi = 0 \; . \tag{13}$$

The lowest eigenvalue in this equation is $\omega^2_{min} = 0$ for all ξ. Thus there are no runaway modes for the wall w_{24} at least in this approximation, and the wall is locally stable at all ξ, unless radiative corrections push the ω^2_{min} into the negative region.

To summarize the discussion of this section: with respect to small perturbations the "diagonal" wall w_{13} is stable at $\xi < 1$ and is unstable at $\xi > 1$, while the other "diagonal" w_{24} is stable for all ξ at least in the approximation considered here.

4 Shapes of the domains: intersections of walls

The relations between the surface energies of the walls determine the shapes of adjacent domains, containing different vacua. One simple example of this has already been mentioned: if the wall w_{13} is unstable ($\xi > 1$), the domains with v_1 and v_3 cannot be adjacent.

There necessarily should be a domain with either v_2 or v_4 separating them. Naturally, in the three dimensional space there arises a whole variety of configurations involving more than two different domains.

Let us first discuss the planar configurations, i.e. those where the fields depend on two spatial variables. In a one-field theory it is known that an intersection of domain walls is unstable. Indeed the configuration shown in Fig. 2a has translational zero modes, whose shape is given by the gradient of the field. Since there are directions in the plane, where the field approaches the same values at both infinities, the component of the gradient in such direction necessarily has zero. Thus the zero mode cannot be the lowest in the spectrum and a negative mode exists, leading to a separation of the walls shown in Fig. 2b. However if, as in supersymmetric theories, there are more than two degenerate vacua, one can easily construct a "triple intersection", as shown in Fig.3. If the energy densities of the three domain walls satisfy the triangle condition, i.e. each of the energies is less than the sum of the other two, one can find the stable shape of such triple intersection. Indeed, the wall acts as a film with the surface tension given by the energy density ε. Thus the relative angles between the walls in the equilibrium shape are determined by the balance of forces. In terms of the notation in Fig. 3 the conditions for this balance are: $\varepsilon_{ac} \sin\alpha = \varepsilon_{bc} \sin\beta$ and $\varepsilon_{ac} \cos\alpha + \varepsilon_{bc} \cos\beta = \varepsilon_a b$. Viewed as equations for the angles α and β these conditions always have a real solution, provided that the three surface tensions satisfy the triangle condition. If the triangle condition is not satisfied, so that e.g. $\varepsilon_{ab} > \varepsilon_{ac} + \varepsilon_{bc}$ the equilibrium is impossible and the configuration stratifies into one where the domain v_c separates the domains v_a and v_b[5].

One can consider, as a logical possibility, an instability of the triple intersection with respect to the process, shown in Fig.4, i.e. when a fourth phase v_d intervenes in the middle, providing a lower energy due to a lower wall tension between the domain v_d and each of the three other domains. However a simple geometric consideration reveals that the energy is lowered only in the case where one of the walls in Fig. 3 is unstable with respect to decay into two walls with the domain of v_d in between. In this case the configuration of Fig. 3 is unphysical since the domain with v_d will intervene anyway.

Consider now a quadruple intersection as shown in Fig. 5a with each of the four walls assumed to be stable. From a simple geometrical counting of the total surface energy one concludes that such intersection should split into two triple intersections, i.e. into the

[5]Note, that in this situation the equilibrium equations would have a solution with imaginary α and β, which would correspond to the rapidities of the walls w_{ac} and w_{bc} in the decay of the wall w_{ab}.

configuration of either Fig. 5b or Fig. 5c, depending on which of the walls w_{ac} or w_{bd} has lower energy and is stable. In the model considered in this paper at least one of the latter walls is stable. It is not known at present, whether this is valid in general case. Therefore one may speculate that it may be that both the w_{ac} and w_{bd} are unstable, and the shapes in both Fig. 5b and Fig. 5c do not exist in equilibrium. Then the quadruple intersection of Fig. 5a has to be stable. In this case the balance of tension forces leaves undetermined one angle parameter. It is quite likely that this parameter is fixed by finer local effects of the precise profiles of the fields at the point of intersection.

In the two-field model considered here, the latter situation is not realized and the quadruple, as well as higher, intersections split into triple ones, which are the simplest non-trivial ones. The same is true for the three dimensional intersections of the domain wall planes in space. The minimal non-trivial intersection is a tetrahedral intersection, where four domains meet at one point, and the relative angles are determined by the balance of the surface tension forces. More complicated intersection will split into ensembles of these simpler ones, provided that the walls formed in the process of splitting are stable.

5 Summary

The presented consideration of the structure of domain walls in a simple two-field model with the superpotential (1) illustrates that the phenomenology of the domain walls in supersymmetric models can be quite feature-rich with a non-trivial dependence on the couplings. Some of the walls can be unstable and decay into pairs of other walls at certain values of the parameters. In the model considered here the observed instability of a wall (w_{13} at $\xi > 1$) is complete, i.e. the field of this configuration develops a runaway mode at the same values of the appropriate coupling constant parameter at which the wall is globally unstable. It is not known at present whether this is a universal behavior or metastable walls can be constructed in more general models. When regarded as candidates for a realistic model the supersymmetric schemes with multiple domain structure can be tested against the dynamics of the early Universe, where domain formation is known to be subject to observational constraints[5]. It should be noticed however, that a consideration of the vacuum domain structure in the early Universe may be complicated by thermal effects, which in principle lift the energy degeneracy between at least some of the vacua. Clearly, a better understanding of the phenomenology of the domain walls both at $T = 0$ as well as at a finite temperature would require a much more extensive study than is

presented in this paper.

6 Acknowledgement

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After this paper was finished, there appeared a paper by M. Shifman[6], where a degenerate class of domain walls in supersymmetric models is considered as well wall stability and instability for a superpotential equivalent to that in eq.(1) discussed here.

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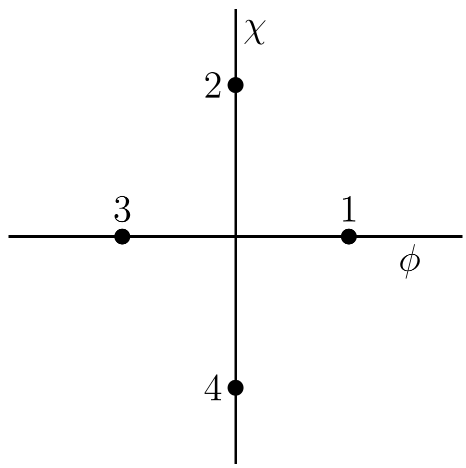

Figure 1: Four supersymmetric vacua in a theory with the superpotential (1).

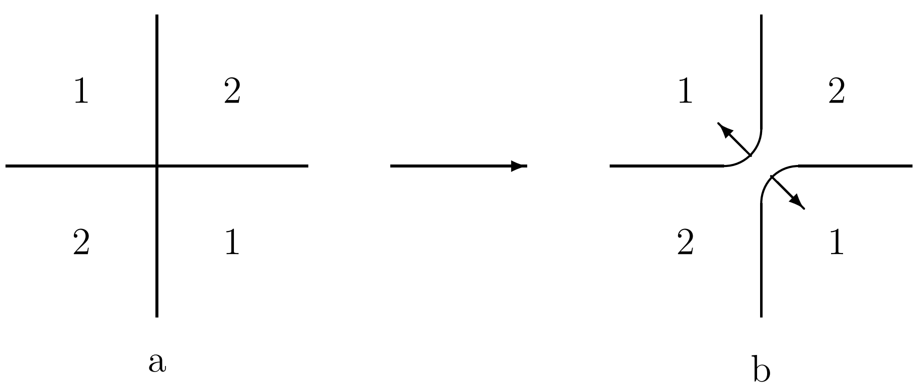

Figure 2: Intersection of domain walls in a one-field theory (a) is unstable and decays into the configuration of the type (b) with the walls moving in opposite directions.

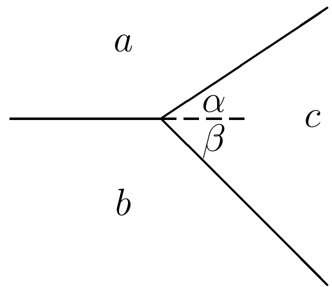

Figure 3: Intersection of three domain walls. The angles α and β are determined by the equillibrium of the tension forces.

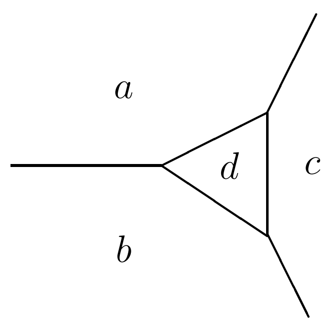

Figure 4: A configuration, whose energy is lower than that of shown in Fig. 3 only if one of the walls in Fig. 3 is unstable.

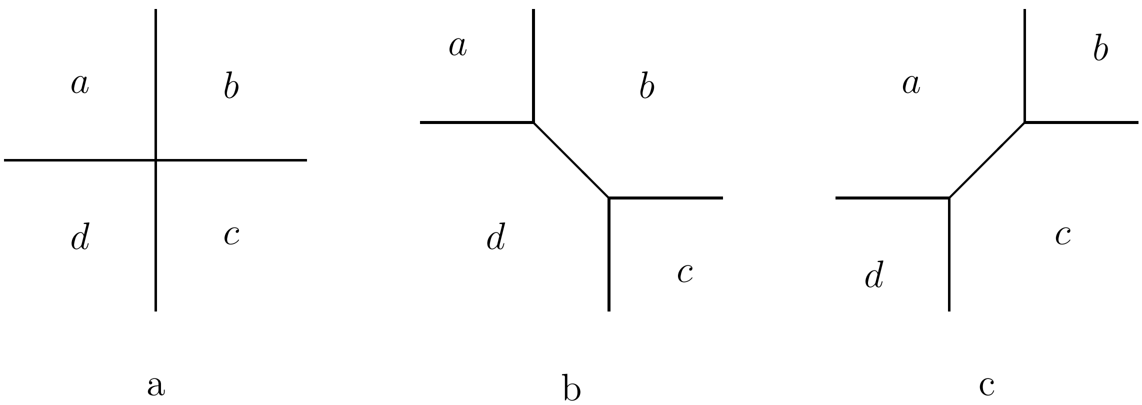

Figure 5: An intersection of four domain walls (a) splits into two triple intersections (the configuration (b) or (c)), provided that at least one of the walls w_{ac} and w_{bd} is stable.